

A variant of Strassen's theorem: Existence of martingales within a prescribed distance

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Abstract

Strassen's theorem asserts that a stochastic process is increasing in convex order if and only if there is a martingale with the same one-dimensional marginal distributions. Such processes, respectively families of measures, are nowadays known as peacocks. We extend this classical result in a novel direction, relaxing the requirement on the martingale. Instead of equal marginal laws, we just require them to be within closed balls, defined by some metric on the space of probability measures. In our main result, the metric is the infinity Wasserstein distance. Existence of a peacock within a prescribed distance is reduced to a countable collection of rather explicit conditions. We also solve this problem when the underlying metric is the stop-loss distance, the Lévy distance, or the Prokhorov distance. This result has a financial application (developed in a separate paper), as it allows to check European call option quotes for consistency. The distance bound on the peacock then takes the role of a bound on the bid-ask spread of the underlying.

1 Introduction

A famous result, first proved by Strassen in 1965,¹ states that, for a given sequence of probability measures $(\mu_n)_{n \in \mathbb{N}}$, there exists a martingale $M = (M_n)_{n \in \mathbb{N}}$ such that the law of M_n is μ_n for all n , if and only if all μ_n have finite mean and $(\mu_n)_{n \in \mathbb{N}}$ is increasing in convex order (see Definition 2.1). Such sequences, and their continuous time counterparts, are nowadays referred to as peacocks, a pun on the french acronym PCOC, for “Processus Croissant pour l'Ordre Convexe” [15]. For further references on Strassen's theorem and its predecessors, see the appendix of [7], p.380 of Dellacherie and Meyer [9], and [1].

The theorem gave rise to plenty of generalizations, one of the most famous being Kellerer's theorem [19, 20]. It states that, for a peacock $(\mu_t)_{t \geq 0}$ with index set \mathbb{R}^+ , there is a *Markov* martingale $M = (M_t)_{t \geq 0}$ such that $M_t \sim \mu_t$ for all $t \geq 0$. Several proofs and ramifications of Kellerer's theorem can be found in the literature. Hirsch and Roynette [16] construct martingales as solutions of stochastic differential equations and use an approximation argument. Lowther [25, 26] shows that under some regularity assumptions there exists an ACD martingale with marginals $(\mu_t)_{t \geq 0}$. Here, ACD stands for “almost-continuous diffusion”, a condition implying the strong

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¹See Theorem 8 in [31]. (Another result from that paper, relative to the usual stochastic order instead of the convex order, is also sometimes referred to as Strassen's theorem; see [23].)

Markov property and stochastic continuity. Beiglböck, Huesmann and Stebegg [2] use a certain solution of the Skorokhod problem, which is Lipschitz-Markov, to construct a martingale which is Markov. The recent book by Hirsch, Profeta, Roynette, and Yor [15] contains a wealth of constructions of peacocks and associated martingales.

While there are many works that aim at producing martingales with additional properties, we extend Strassen's theorem in a different direction. The main question that we consider in this paper is the following: given $\epsilon > 0$, a metric d on \mathcal{M} – the set of all probability measures on \mathbb{R} with finite mean – and a sequence of measures $(\mu_t)_{t \in T}$ in \mathcal{M} , when does a sequence $(\nu_t)_{t \in T}$ in \mathcal{M} exist, such that $d(\mu_t, \nu_t) \leq \epsilon$ and such that the sequence $(\nu_t)_{t \in T}$ is a peacock? Here T is either \mathbb{N} or the interval $[0, 1]$. Once we have constructed a peacock, we know, from the results mentioned above, that there is a martingale (with certain properties) with these marginals. We thus want to find out when there is a martingale M such that the law of M_t is close to μ_t for all t . We will state necessary and sufficient conditions when d is the infinity Wasserstein distance, the stop-loss distance, the Prokhorov distance, and the Lévy distance.

The infinity Wasserstein distance is a natural analogue of the well-known p -Wasserstein distance. It seems to have made only a few appearances in the literature, one being [6], where the authors study it in an optimal transport setting. It also has applications in graph theory, where it is referred to as the bottleneck distance (see p. 216 of [11]). We will give an alternative representation of the infinity Wasserstein distance, which shows some similarity to the better known Lévy distance. The stop-loss distance was introduced by Gerber in [12] and has been studied in actuarial science (see for instance [8, 18]).

For both of these metrics, we translate existence of a peacock within ϵ -distance into a more tractable condition: There has to exist a real number (with the interpretation of the desired peacock's mean) that satisfies a countable collection of finite-dimensional conditions, each explicitly expressed in terms of the call functions $x \mapsto \int (y - x)^+ \mu_t(dy)$ of the given sequence of measures. For the infinity Wasserstein distance, the existence proof is not constructive, as it uses Zorn's lemma. For the stop-loss distance, the problem is much simpler, and our proof is short and constructive. Note, though, that the result about the infinity Wasserstein distance admits a financial application, which was the initial motivation for this work. The problem is similar to the one considered by Davis and Hobson [7]: given a set of European call option prices with different maturities on one underlying, we want to know when there is a model which is consistent with these prices. In contrast to Davis and Hobson we allow a bid-ask spread, bounded by some constant, on the underlying. This application will be developed in the companion paper [13].

Our proof approach is similar for both metrics: we will construct minimal and maximal elements (with respect to convex ordering) in closed balls, and then use these elements to derive our conditions. In the case of the infinity Wasserstein distance, we will make use of the lattice structure of certain subsets of closed balls.

The Lévy distance was first introduced by Lévy in 1925 (see [22]). Its importance is partially due to the fact that d^L metrizes weak convergence of measures on \mathbb{R} . The Prokhorov distance, first introduced in [29], is a metric on measures on an arbitrary separable metric space, and is often referred to as a generalization of the Lévy metric, since d^P metrizes weak convergence on any separable metric space. For these two metrics, peacocks within ϵ -distance *always* exist, and can be explicitly constructed.

The structure of the paper is as follows. Section 2 specifies our notation and introduces the most important definitions. In Section 3 contains our main results, concerning the described variant of Strassen's theorem for the infinity Wasserstein distance. A continuous time version of this can be found in Section 4. In Section 5 we will treat the stop-loss distance. After collecting some facts on the Lévy and Prokhorov distances in Section 6, we will prove a variant of Strassen's theorem for these metrics in Sections 7-8.

2 Notation and preliminaries

Let \mathcal{M} denote the set of all probability measures on \mathbb{R} with finite mean. We start with the definition of convex order.

Definition 2.1. Let μ, ν be two measures in \mathcal{M} . Then we say that μ is smaller in convex order than ν , in symbols $\mu \leq_c \nu$, if for every convex function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ we have that $\int \phi d\mu \leq \int \phi d\nu$, whenever both integrals are finite.² A family of measures $(\mu_t)_{t \in T}$ in \mathcal{M} , where $T \subseteq [0, \infty)$, is called peacock, if $\mu_s \leq_c \mu_t$ for all $s \leq t$ in T (see Definition 1.3 in [15]).

Intuitively, $\mu \leq_c \nu$ means that ν is more dispersed than μ , as convex integrands tend to emphasize the tails. By choosing $\phi(x) = x$ resp. $\phi(x) = -x$, we see that $\mu \leq_c \nu$ implies that μ and ν have the same mean. As mentioned in the introduction, Strassen's theorem asserts that, for any peacock, there is a martingale whose family of marginal laws coincides with it; the converse is a trivial consequence of Jensen's inequality. For $\mu \in \mathcal{M}$ and $x \in \mathbb{R}$ we define

$$R_\mu(x) = \int_{\mathbb{R}} (y - x)^+ \mu(dy) \quad \text{and} \quad F_\mu(x) = \mu((-\infty, x]).$$

We call R_μ the call function of μ , as in financial terms it is the (undiscounted) price of a call option with strike x , written on an underlying with law μ at maturity. The mean of a measure μ will be denoted by $\mathbb{E}\mu = \int y \mu(dy)$. The following proposition summarizes important properties of call functions.

Proposition 2.2. *Let μ, ν be two measures in \mathcal{M} . Then:*

- (i) R_μ is convex, decreasing and strictly decreasing on $\{R_\mu > 0\}$. Hence the right derivative of R_μ always exists and is denoted with R'_μ .
- (ii) $\lim_{x \rightarrow \infty} R_\mu(x) = 0$ and $\lim_{x \rightarrow -\infty} (R_\mu(x) + x) = \mathbb{E}\mu$. In particular, if $\mu([a, \infty)) = 1$ for $a > -\infty$, then $\mathbb{E}\mu = R_\mu(a) + a$.
- (iii) $R'_\mu(x) = -1 + F_\mu(x)$ and $R_\mu(x) = \int_x^\infty (1 - F_\mu(y)) dy$, for all $x \in \mathbb{R}$.
- (iv) $\mu \leq_c \nu$ holds if and only if $R_\mu(x) \leq R_\nu(x)$ for all $x \in \mathbb{R}$ and $\mathbb{E}\mu = \mathbb{E}\nu$.
- (v) For $x_1 \leq x_2 \in \mathbb{R}$, we have $R_\mu(x_2) - R_\mu(x_1) = \int_{x_1}^{x_2} R'_\mu(y) dy$.

Conversely, if a function $R : \mathbb{R} \rightarrow \mathbb{R}$ satisfies (i) and (ii), then there exists a probability measure $\mu \in \mathcal{M}$ with finite mean such that $R_\mu = R$.

Proof. As for (v), note that R'_μ is increasing, thus integrable, and that the fundamental theorem of calculus holds for right derivatives. See [4] for a short proof. The other assertions are proved in [16], Proposition 2.1, and [15], Exercise 1.7. \square

For a metric d on \mathcal{M} , denote with $B^d(\mu, \epsilon)$ the closed ball with respect to d , with center μ and diameter ϵ . Then our main question is:

²The apparently stronger requirement that the inequality $\int \phi d\mu \leq \int \phi d\nu$ holds for convex ϕ whenever it makes sense, i.e., as long as both sides exist in $[-\infty, \infty]$, leads to an equivalent definition. This can be seen by the following argument, similar to Remark 1.1 in [15]: Assume that the inequality holds if both sides are finite, and let ϕ (convex) be such that $\int \phi d\mu = \infty$. We have to show that then $\int \phi d\nu = \infty$. Since ϕ is the envelope of the affine functions it dominates, we can find convex ϕ_n with $\phi_n \uparrow \phi$ pointwise, and such that each ϕ_n is C^2 and ϕ_n'' has compact support. By monotone convergence, we then have $\int \phi d\nu = \lim \int \phi_n d\nu \geq \lim \int \phi_n d\mu = \int \phi d\mu = \infty$. With similar arguments we can deal with the case where $\int \phi d\nu = -\infty$.

Problem 2.3. Given $\epsilon > 0$, a metric d on \mathcal{M} , and a sequence $(\mu_n)_{n \in \mathbb{N}}$ in \mathcal{M} , when does there exist a peacock $(\nu_n)_{n \in \mathbb{N}}$ with $\nu_n \in B^d(\mu_n, \epsilon)$ for all n ?

Note that this can also be phrased as

$$d_\infty((\mu_n)_{n \in \mathbb{N}}, (\nu_n)_{n \in \mathbb{N}}) \leq \epsilon,$$

where

$$d_\infty((\mu_n)_{n \in \mathbb{N}}, (\nu_n)_{n \in \mathbb{N}}) = \sup_{n \in \mathbb{N}} d(\mu_n, \nu_n)$$

defines a metric on $\mathcal{M}^\mathbb{N}$ (with possible value infinity; see the remark before Proposition 2.5 below). For some results on this kind of infinite product metric, we refer to [3].

To fix ideas, consider the case where the given sequence $(\mu_n)_{n=1,2}$ has only two elements. We want to find measures $\nu_n \in B^d(\mu_n, \epsilon)$, $n = 1, 2$, such that $\nu_1 \leq_c \nu_2$. Intuitively, we want ν_1 to be as small as possible and ν_2 to be as large as possible, in the convex order. Recall that a peacock has constant mean, which is fixed as soon as ν_1 is chosen. We will denote the set of probability measures on \mathbb{R} with mean $m \in \mathbb{R}$ by \mathcal{M}_m . These considerations lead us to the following problem.

Problem 2.4. Suppose that a metric d on \mathcal{M} , a measure $\mu \in \mathcal{M}$ and two positive numbers ϵ, m are given. When are there two measures $\mu^{\min}, \mu^{\max} \in B^d(\mu, \epsilon) \cap \mathcal{M}_m$ such that

$$\mu^{\min} \leq_c \nu \leq_c \mu^{\max}, \quad \text{for all } \nu \in B^d(\mu, \epsilon) \cap \mathcal{M}_m ?$$

The following proposition defines the infinity Wasserstein distance³ W^∞ , and explains its connection to call functions. For various other probability metrics and their relations, see [14]. We will use the words “metric” and “distance” for mappings $\mathcal{M} \times \mathcal{M} \rightarrow [0, \infty]$ in a loose sense. Since all our results concern *concrete* metrics, there is no need to give a general definition (as, e.g., Definition 1 in Zolotarev [32]). For the sake of completeness, we include a proof that W^∞ satisfies the classical properties of a metric. Note also that allowing metrics to take the value ∞ , as we do, leaves much of the theory of metric spaces unchanged; see, e.g., [5].

Proposition 2.5. *The mapping $W^\infty : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty]$, defined by*

$$W^\infty(\mu, \nu) = \inf \|X - Y\|_\infty,$$

satisfies the metric axioms. The infimum is taken over all probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and random pairs (X, Y) with marginals given by μ and ν . This metric has the following representation in terms of call functions, which is more useful for our purposes:

$$W^\infty(\mu, \nu) = \inf \left\{ h > 0 : R'_\mu(x - h) \leq R'_\nu(x) \leq R'_\mu(x + h), \forall x \in \mathbb{R} \right\}. \quad (2.1)$$

Proof. For the equivalence of the two representations see [24], p. 127. Clearly, W^∞ is symmetric and $W^\infty(\mu, \mu) = 0$. If we assume that $W^\infty(\mu, \nu) = 0$, then we have for each $n \in \mathbb{N}$ and $x \in \mathbb{R}$

$$R'_\mu\left(x - \frac{1}{n}\right) \leq R'_\nu(x) \leq R'_\mu\left(x + \frac{1}{n}\right),$$

and hence $R'_\nu(x) \leq R'_\mu(x)$. By symmetry, we get $R'_\mu(x) \leq R'_\nu(x)$, which implies that $R_\mu = R_\nu$ and hence $\mu = \nu$.

³The name “*infinite* Wasserstein distance” is also in use, but “*infinity* Wasserstein distance” seems to make more sense (cf. “*infinity* norm”).

Given three measures $\mu_1, \mu_2, \mu_3 \in \mathcal{M}$ with $W^\infty(\mu_1, \mu_2) = \epsilon_1 < \infty$ and $W^\infty(\mu_2, \mu_3) = \epsilon_2 < \infty$ we obtain that

$$\begin{aligned} R'_{\mu_1}\left(x - \left(\epsilon_1 + \epsilon_2 + \frac{2}{n}\right)\right) &\leq R'_{\mu_2}\left(x - \left(\epsilon_2 + \frac{1}{n}\right)\right) \\ &\leq R'_{\mu_3}(x) \leq R'_{\mu_2}\left(x + \left(\epsilon_2 + \frac{1}{n}\right)\right) \\ &\leq R'_{\mu_1}\left(x + \left(\epsilon_1 + \epsilon_2 + \frac{2}{n}\right)\right). \end{aligned}$$

Thus

$$W^\infty(\mu_1, \mu_3) \leq \epsilon_1 + \epsilon_2 = W^\infty(\mu_1, \mu_2) + W^\infty(\mu_2, \mu_3).$$

Note that the triangle-inequality trivially holds if $\max\{\epsilon_1, \epsilon_2\} = \infty$. \square

By (2.1) and Proposition 2.2 (iii), W^∞ can also be written as

$$W^\infty(\mu, \nu) = \inf\left\{\epsilon > 0 : F_\mu(x - \epsilon) \leq F_\nu(x) \leq F_\mu(x + \epsilon), \forall x \in \mathbb{R}\right\}.$$

We will see below (Proposition 3.2) that, when d is the infinity Wasserstein distance, Problem 2.4 has a solution (μ^{\min}, μ^{\max}) if and only if $|m - \mathbb{E}\mu| \leq \epsilon$. As an easy consequence, given $(\mu_n)_{n=1,2}$, the desired “close” peacock $(\nu_n)_{n=1,2}$ exists if and only if there is an m with $|m - \mathbb{E}\mu_1| \leq \epsilon$, $|m - \mathbb{E}\mu_2| \leq \epsilon$ such that the corresponding measures $\mu_1^{\min}, \mu_2^{\max}$ satisfy $\mu_1^{\min} \leq_c \mu_2^{\max}$. Then, $(\nu_1, \nu_2) = (\mu_1^{\min}, \mu_2^{\max})$ is a possible choice.

Besides the infinity Wasserstein distance, we will solve Problems 2.3 and 2.4 also for the stop-loss distance (Proposition 5.1), for index sets \mathbb{N} and $[0, 1]$ (see Theorems 3.5, 4.1, 5.3, and 5.5). For the Lévy distance and the Prokhorov distance we will use different techniques and solve Problem 2.3 for index set \mathbb{N} (see Corollary 8.4 and Theorem 8.5).

3 Strassen’s theorem for the infinity Wasserstein distance: discrete time

We now start to investigate the interplay between the infinity Wasserstein distance and the convex order. It is a well known fact that the ordered set (\mathcal{M}_m, \leq_c) is a lattice for all $m \in \mathbb{R}$, with least element δ_m (Dirac delta). See for instance [21, 27]; recall that \mathcal{M}_m denotes the set of probability measures on \mathbb{R} with mean m . The lattice property means that, given any two measures $\mu, \nu \in \mathcal{M}_m$, there is a unique supremum, denoted with $\mu \vee \nu$, and a unique infimum, denoted with $\mu \wedge \nu$, with respect to convex order. It is easy to prove that $R_{\mu \vee \nu} = R_\mu \vee R_\nu$ and $R_{\mu \wedge \nu} = \text{conv}(R_\mu, R_\nu)$. Here and in the following $\text{conv}(R_\mu, R_\nu)$ denotes the convex hull of R_μ and R_ν , i.e., the largest convex function that is majorized by $R_\mu \wedge R_\nu$.

In the following we will denote balls with respect to W^∞ with B^∞ . The next lemma shows that $(B^\infty(\mu, \epsilon) \cap \mathcal{M}_m, \leq_c)$ is a sublattice of (\mathcal{M}_m, \leq_c) , which will be important afterwards. Recall that two measures can be comparable w.r.t. convex order only if their means agree. This accounts for the relevance of sublattices of the form $(B^\infty(\mu, \epsilon) \cap \mathcal{M}_m, \leq_c)$ for our problem: If a peacock $(\nu_n)_{n \in \mathbb{N}}$ satisfying $\nu_n \in B^\infty(\mu_n, \epsilon)$ for all $n \in \mathbb{N}$ exists, then we necessarily have $\nu_n \in B^\infty(\mu_n, \epsilon) \cap \mathcal{M}_m$, $n \in \mathbb{N}$, with $\mathbb{E}\nu_1 = \mathbb{E}\nu_2 = \dots = m$.

Lemma 3.1. *Let $\mu \in \mathcal{M}$, $\epsilon > 0$ and $m \in \mathbb{R}$. Then for $\nu_1, \nu_2 \in B^\infty(\mu, \epsilon) \cap \mathcal{M}_m$ we have $\nu_1 \vee \nu_2 \in B^\infty(\mu, \epsilon) \cap \mathcal{M}_m$ and $\nu_1 \wedge \nu_2 \in B^\infty(\mu, \epsilon) \cap \mathcal{M}_m$.*

Proof. Denote the call functions of ν_1 and ν_2 with R_1 and R_2 . We start with $\nu_1 \vee \nu_2$. It is easy to check that $R : x \mapsto R_1(x) \vee R_2(x)$ is a call function such that $R'(x) \in \{R'_1(x), R'_2(x)\}$ for all $x \in \mathbb{R}$. By Proposition 2.2 (ii), it is also clear that $\nu_1 \vee \nu_2 \in \mathcal{M}_m$. This proves the assertion.

As for the infimum, we will first assume that there exists $x_0 \in \mathbb{R}$ such that $R_1(x) \leq R_2(x)$ for $x \leq x_0$ and $R_2(x) \leq R_1(x)$ for $x \geq x_0$. Then there exist $x_1 \leq x_0$ and $x_2 \geq x_0$ such that the convex hull of R_1 and R_2 can be written as (see [28])

$$\text{conv}(R_1, R_2)(x) = \begin{cases} R_1(x), & x \leq x_1, \\ R_1(x_1) + \frac{R_2(x_2) - R_1(x_1)}{x_2 - x_1}(x - x_1), & x \in [x_1, x_2], \\ R_2(x), & x \geq x_2. \end{cases}$$

Now observe that for all $x \in [x_1, x_2]$

$$\begin{aligned} R'_\mu(x - \epsilon) &\leq R'_2(x) \leq R'_2(x_2 -) \\ &\leq \frac{R_2(x_2) - R_1(x_1)}{x_2 - x_1} \\ &\leq R'_1(x_1) \leq R'_1(x) \leq R'_\mu(x + \epsilon), \end{aligned}$$

and hence $\text{conv}(R_1, R_2)'(x) \in [R'_\mu(x - \epsilon), R'_\mu(x + \epsilon)]$. Therefore $\nu_1 \wedge \nu_2 \in B^\infty(\mu, \epsilon) \cap \mathcal{M}_m$.

For the general case note that for all $x \in \mathbb{R}$ we have by [28] that either $\text{conv}(R_1, R_2)(x) = R_\mu(x) \wedge R_\nu(x)$ or that x lies in an interval I such that $\text{conv}(R_1, R_2)$ is affine on I . If the latter condition is the case then we can derive bounds for the right-derivative $\text{conv}(R_1, R_2)'(x), x \in I$, exactly as before. The situation is clear if either $\text{conv}(R_1, R_2)(x) = R_1(x)$ or $\text{conv}(R_1, R_2)(x) = R_2(x)$. \square

We now show that the sublattice $(B^\infty(\mu, \epsilon) \cap \mathcal{M}_m, \leq_c)$ contains a least and a greatest element with respect to convex order. This is the subject of the following proposition, and is also the solution to Problem 2.4 for the infinity Wasserstein distance. As for the assumption $m \in [\mathbb{E}\mu - \epsilon, \mathbb{E}\mu + \epsilon]$ in Proposition 3.2, it is necessary to ensure that $B^\infty(\mu, \epsilon) \cap \mathcal{M}_m$ is not empty. Indeed, if $W^\infty(\mu_1, \mu_2) \leq \epsilon$ for some $\mu_1, \mu_2 \in \mathcal{M}$, then by (2.1), Proposition 2.2 (ii), (v), and the continuity of call functions, we obtain

$$R_{\mu_1}(x + \epsilon) \leq R_{\mu_2}(x) \leq R_{\mu_1}(x - \epsilon), \quad x \in \mathbb{R}. \quad (3.1)$$

By part (ii) of Proposition 2.2, it follows that $|\mathbb{E}\mu_1 - \mathbb{E}\mu_2| \leq \epsilon$.

Proposition 3.2. *Given $\epsilon > 0$, a measure $\mu \in \mathcal{M}$ and $m \in [\mathbb{E}\mu - \epsilon, \mathbb{E}\mu + \epsilon]$, there exist unique measures $S(\mu), T(\mu) \in B^\infty(\mu, \epsilon) \cap \mathcal{M}_m$ such that*

$$S(\mu) \leq_c \nu \leq_c T(\mu) \quad \text{for all } \nu \in B^\infty(\mu, \epsilon) \cap \mathcal{M}_m.$$

The call functions of $S(\mu)$ and $T(\mu)$ are given by

$$R_\mu^{\min}(x) = R_{S(\mu)}(x) = \left(m + R_\mu(x - \epsilon) - (\mathbb{E}\mu + \epsilon) \right) \vee R_\mu(x + \epsilon), \quad (3.2)$$

$$R_\mu^{\max}(x) = R_{T(\mu)}(x) = \text{conv}\left(m + R_\mu(\cdot + \epsilon) - (\mathbb{E}\mu - \epsilon), R_\mu(\cdot - \epsilon) \right)(x). \quad (3.3)$$

To highlight the dependence on ϵ and m we will sometimes write $S(\mu; m, \epsilon)$ and $R_\mu^{\min}(\cdot; m, \epsilon)$, respectively $T(\mu; m, \epsilon)$ and $R_\mu^{\max}(\cdot; m, \epsilon)$.

Proof. We define R_μ^{\min} and R_μ^{\max} by the right hand sides of (3.2) resp. (3.3), and argue that the associated measures $S(\mu)$ and $T(\mu)$ have the stated property. Clearly R_μ^{\min} is a call function, and we have that

$$\begin{aligned}\mathbb{E}R_\mu^{\min} &= \lim_{x \rightarrow -\infty} \left(m + R_\mu(x - \epsilon) - (\mathbb{E}\mu + \epsilon) + x \right) \vee \left(R_\mu(x + \epsilon) + x \right) \\ &= m \vee (\mathbb{E}\mu - \epsilon) = m.\end{aligned}$$

From the convexity of R_μ we can deduce the existence of $v \in \mathbb{R} \cup \{\pm\infty\}$ such that

$$R_\mu^{\min}(x) = \begin{cases} m + R_\mu(x - \epsilon) - (\mathbb{E}R_\mu + \epsilon), & x \leq v, \\ R_\mu(x + \epsilon) & x \geq v. \end{cases}$$

Hence we get that $(R_\mu^{\min})'(x) \in [R'_\mu(x - \epsilon), R'_\mu(x + \epsilon)]$ for all x . According to Proposition 2.5, the measure associated with R_μ^{\min} lies in $B^\infty(\mu, \epsilon) \cap \mathcal{M}_m$. To the left of v , R_μ^{\min} is as steep as possible (where steepness refers to the absolute value of the right derivative), and to the right of v it is as flat as possible. From this and convexity, it is easy to see that $S(\mu)$ is a least element.

Similarly we can show that $\mathbb{E}R_\mu^{\max} = m$, and thus it suffices to show that

$$(R_\mu^{\max})'(x) \in [R'_\mu(x - \epsilon), R'_\mu(x + \epsilon)].$$

But this can be done exactly as in Lemma 3.1. □

Remark 3.3. It is not hard to show that

$$R_\mu^{\max}(x) = \begin{cases} m + R_\mu(x + \epsilon) - (\mathbb{E}\mu - \epsilon), & x \leq x_1, \\ R_\mu(x_1 + \epsilon) + \frac{(\mathbb{E}\mu - \epsilon) - m}{2\epsilon}(x - x_1 - 2\epsilon), & x \in [x_1, x_1 + 2\epsilon], \\ R_\mu(x - \epsilon), & x \geq x_1 + 2\epsilon, \end{cases}$$

where

$$x_1 = \inf \left\{ x \in \mathbb{R} \mid R'_\mu(x + \epsilon) \geq -\frac{m - (\mathbb{E}\mu - \epsilon)}{2\epsilon} \right\}.$$

Before formulating our main theorem, we recall that in Definition 2.1 we defined a peacock to be a sequence of probability measures with finite mean and increasing w.r.t. convex order. We now give a simple reformulation of this property. For a given sequence of call functions $(R_n)_{n \in \mathbb{N}}$, define, for $N \in \mathbb{N}$ and $x_1, \dots, x_N \in \mathbb{R}$,

$$\Phi_N(x_1, \dots, x_N) = R_1(x_1) + \sum_{n=2}^N \left(R_n(x_n) - R_n(x_{n-1}) \right) - R_{N+1}(x_N). \quad (3.4)$$

Proposition 3.4. *A sequence of call functions $(R_n)_{n \in \mathbb{N}}$ with constant mean defines a peacock if and only if $\Phi_N(x_1, \dots, x_N) \leq 0$ for all $N \in \mathbb{N}$ and $x_1, \dots, x_N \in \mathbb{R}$.*

Proof. According to Proposition 2.2 (iv), we need to check whether the sequence of call functions increases. Let $n \in \mathbb{N}$ be arbitrary. If we set the n -th component of (x_1, \dots, x_{n+1}) to an arbitrary $x \in \mathbb{R}$ and let all others tend to ∞ , we get

$$\Phi_{n+1}(\infty, \dots, \infty, x, \infty) = R_n(x) - R_{n+1}(x).$$

The sequence of call functions thus increases, if Φ is always non-positive. Conversely, assume that $(R_n)_{n \in \mathbb{N}}$ increases. Then, for $N \in \mathbb{N}$ and $x_1, \dots, x_N \in \mathbb{R}$,

$$\begin{aligned}\Phi_N(x_1, \dots, x_N) &\leq R_1(x_1) + \sum_{n=2}^N R_{n+1}(x_n) - \sum_{n=2}^N R_n(x_{n-1}) - R_{N+1}(x_N) \\ &= R_1(x_1) + \sum_{n=3}^{N+1} R_n(x_{n-1}) - \sum_{n=2}^N R_n(x_{n-1}) - R_{N+1}(x_N) \\ &= R_1(x_1) - R_2(x_1) \leq 0.\end{aligned}$$

□

We now extend the definition of Φ_N for $x_1, \dots, x_N \in \mathbb{R}$, $m \in \mathbb{R}$, and $\epsilon > 0$ as follows, using the notation from Proposition 3.2:

$$\begin{aligned}\Phi_N(x_1, \dots, x_N; m, \epsilon) &= R_1^{\min}(x_1; m, \epsilon) \\ &\quad + \sum_{n=2}^N \left(R_n(x_n + \epsilon \sigma_n) - R_n(x_{n-1} + \epsilon \sigma_n) \right) - R_{N+1}^{\max}(x_N; m, \epsilon).\end{aligned}\quad (3.5)$$

Here, R_1^{\min} is the call function of $S(\mu_1; m, \epsilon)$, R_{N+1}^{\max} is the call function of $T(\mu_{N+1}; m, \epsilon)$, and

$$\sigma_n = \text{sgn}(x_{n-1} - x_n) \quad (3.6)$$

depends on x_{n-1} and x_n . Clearly, for $\epsilon = 0$ and $\mathbb{E}\mu_1 = \mathbb{E}\mu_2 = \dots = m$, we recover (3.4):

$$\Phi_N(x_1, \dots, x_N; m, 0) = \Phi_N(x_1, \dots, x_N), \quad N \in \mathbb{N}, \quad x_1, \dots, x_N \in \mathbb{R}. \quad (3.7)$$

The following theorem gives an equivalent condition for the existence of a peacock within W^∞ -distance ϵ of a given sequence of measures, thus solving Problem 2.3 for the infinity Wasserstein distance, and is our main result. By Proposition 3.4 and (3.7), it is consistent with Strassen's theorem, i.e., the case $\epsilon = 0$. Also, note that the functions Φ_N defined in (3.5) have explicit expressions in terms of the given call functions, as R^{\min} and R^{\max} are explicitly given by (3.2) and (3.3). The existence criterion we obtain is thus rather explicit; the existence *proof* is not constructive, though, as mentioned in the introduction. Moreover, note that we use Strassen's theorem in the proof; for $\epsilon = 0$, our proof reduces to a triviality, and not to a proof of Strassen's theorem.

Theorem 3.5. *Let $\epsilon > 0$ and $(\mu_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{M} such that*

$$I := \bigcap_{n \in \mathbb{N}} [\mathbb{E}\mu_n - \epsilon, \mathbb{E}\mu_n + \epsilon]$$

is not empty. Denote with $(R_n)_{n \in \mathbb{N}}$ the corresponding call functions, and define Φ_N by (3.5). Then there exists a peacock $(\nu_n)_{n \in \mathbb{N}}$ such that

$$W^\infty(\mu_n, \nu_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}, \quad (3.8)$$

if and only if for some $m \in I$ and for all $N \in \mathbb{N}$, $x_1, \dots, x_N \in \mathbb{R}$, we have

$$\Phi_N(x_1, \dots, x_N; m, \epsilon) \leq 0. \quad (3.9)$$

In this case it is possible to choose $\mathbb{E}\nu_1 = \mathbb{E}\nu_2 = \dots = m$.

The proof of Theorem 3.5 is given towards the end of the present section, building on Theorem 3.8 and Corollary 3.9 below.

For $\epsilon = 0$, condition (3.9) is equivalent to the sequence of call functions (R_n) being increasing, see Proposition 3.4. For $\epsilon > 0$, analogously to the proof of Proposition 3.4, we see that (3.9) implies

$$R_n(x + \epsilon) \leq R_{n+1}(x - \epsilon), \quad x \in \mathbb{R}, n \in \mathbb{N}. \quad (3.10)$$

It is clear that (3.10) is necessary for the existence of the peacock $(\nu_n)_{n \in \mathbb{N}}$, since, by (3.1) and Proposition 2.2 (iv),

$$R_n(x + \epsilon) \leq R_{\nu_n}(x) \leq R_{\nu_{n+1}}(x) \leq R_{n+1}(x - \epsilon), \quad x \in \mathbb{R}, n \in \mathbb{N}.$$

On the other hand, it is easy to show that (3.10) is not sufficient for (3.9):

Example 3.6. Fix $m > 1$ and $\epsilon = 1$ and define two measures

$$\mu_1 = \frac{2}{m+1}\delta_0 + \frac{m-1}{m+1}\delta_{m+1}, \quad \mu_2 = \delta_{m+1},$$

where δ denotes the Dirac delta. It is simple to check that (3.10) is satisfied, i.e.

$$R_{\mu_1}(x + \epsilon) \leq R_{\mu_2}(x - \epsilon), \quad x \in \mathbb{R}.$$

Now assume that we want to construct a peacock $(\nu_n)_{n=1,2}$ such that $W^\infty(\mu_n, \nu_n) \leq 1$. Then the only possible mean for this peacock is m , since $\mathbb{E}\mu_1 = m-1$ and $\mathbb{E}\mu_2 = m+1$ (see the remark before Proposition 3.2). Therefore the peacock has to satisfy $\nu_n \in B^\infty(\mu_n, 1) \cap \mathcal{M}_m$, $n = 1, 2$, and the only possible choice is

$$\nu_1 = \frac{2}{m+1}\delta_1 + \frac{m-1}{m+1}\delta_{m+2}, \quad \nu_2 = \delta_m.$$

But since $R_{\nu_1}(x) > R_{\nu_2}(x)$ for $x \in (1, m+2)$, $(\nu_n)_{n=1,2}$ is not a peacock; see Figure 1.

If the sequence $(\mu_n)_{n=1,2}$ has just two elements, then it suffices to require (3.9) only for $N = 1$. It then simply states that there is an $m \in I$ such that $R_1^{\min}(x; m, \epsilon) \leq R_2^{\max}(x; m, \epsilon)$ for all x , which is clearly necessary and sufficient for the existence of $(\nu_n)_{n=1,2}$.

Example 3.7. Unsurprisingly, the peacock from Theorem 3.5 is in general not unique: Let $\epsilon > 0$ and consider the constant sequences $R_n(x) = (-x)^+$, $n \in \mathbb{N}$, and

$$P_n(x, c) = \begin{cases} -x, & x \leq -\epsilon, \\ \epsilon - \frac{\epsilon(x+\epsilon)}{c+\epsilon}, & -\epsilon \leq x \leq c, \\ 0, & x \geq c. \end{cases}$$

Then, for any $c \in [0, \epsilon]$, it is easy to verify that the sequence of call functions $P_n(\cdot, c)$ defines a peacock satisfying (3.8).

The following theorem furnishes the main step for the induction proof of Theorem 3.5, given at the end of the present section. In each induction step, the next element of the desired peacock should be contained in a certain ball, it should be larger in convex order than the previous element (ν in Theorem 3.8), and it should be as small as possible in order not to hamper the existence of the subsequent elements. This leads us to search for a least element of the set (3.11). The conditions defining this least element translate into inequalities on the corresponding call function. Part (ii) of Theorem 3.8 states that, at each point of the real line, at least one of the latter conditions becomes an equality.

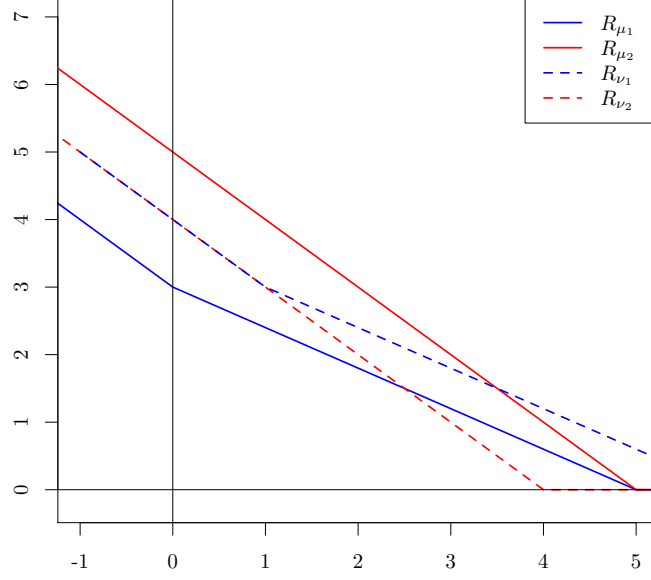


Figure 1: The call functions of μ_1 (lower solid curve) and μ_2 (upper solid curve) from Example 3.6, for $m = 4$ and $\epsilon = 1$. The call function of ν_1 is the call function of μ_1 shifted to the right by one. Similarly, shifting the call function of μ_2 by one to the left yields the call function of ν_2 .

Theorem 3.8. *Let μ, ν be two measures in \mathcal{M} such that the set*

$$A_\mu^\nu := \left\{ \theta \in B^\infty(\mu; \epsilon) : \nu \leq_c \theta \right\} \quad (3.11)$$

is not empty.

- (i) *The set A_μ^ν contains a least element $S_\nu(\mu)$ with respect to \leq_c , i.e. for every $\theta \in A_\mu^\nu$ we have that*

$$\nu \leq_c S_\nu(\mu) \leq_c \theta.$$

Equivalently, if

$$R_\nu(x) \leq R_{T(\mu)}(x; \mathbb{E}\nu, \epsilon), \quad x \in \mathbb{R},$$

there exists a pointwise smallest call function R^ which is greater than R_ν and satisfies $(R^*)'(x) \in [R'_\mu(x - \epsilon), R'_\mu(x + \epsilon)]$ for all $x \in \mathbb{R}$.*

- (ii) *The call function R^* is a solution of the following variational type inequality:*

$$\min \left\{ R^*(x) - R_\nu(x), (R^*)'(x) - R'_\mu(x - \epsilon), R'_\mu(x + \epsilon) - (R^*)'(x) \right\} = 0, \quad x \in \mathbb{R}. \quad (3.12)$$

Proof. The equivalence in (i) follows from Proposition 2.2 (iv). We now argue that $S_\nu(\mu)$ exists. An easy application of Zorn's lemma shows that there exist minimal elements in A_μ^ν . If θ_1 and θ_2 are two minimal elements of A_μ^ν then according to Lemma 3.1 the measure $\theta_1 \wedge \theta_2$ lies in $B^\infty(\nu, \epsilon) \cap \mathcal{M}_{\mathbb{E}\nu}$. Moreover, the convex function R_ν nowhere exceeds R_{θ_1} and R_{θ_2} , and hence we have that $R_\nu \leq \text{conv}(R_{\theta_1} \wedge R_{\theta_2}) = R_{\theta_1 \wedge \theta_2}$. Therefore $\theta_1 \wedge \theta_2$ lies in A_μ^ν . Now clearly $\theta_1 \wedge \theta_2 \leq_c \theta_1$ and $\theta_1 \wedge \theta_2 \leq_c \theta_2$, and from the minimality we can conclude that $\theta_1 \wedge \theta_2 = \theta_1 = \theta_2$.

Now let θ^* be the unique minimal element and let $\theta \in A_\mu^\nu$ be arbitrary. Exactly as before we can show that $\theta^* \wedge \theta$ lies in A_μ^ν . Moreover $\theta^* = \theta^* \wedge \theta \leq_c \theta$ and therefore θ^* is the least element of A_μ^ν .

It remains to show (ii). We set

$$R^*(x) = \inf \{ R_\theta(x) : \theta \in A_\mu^\nu \}. \quad (3.13)$$

Clearly R^* is a decreasing function with $\lim_{x \rightarrow \infty} R^*(x) = 0$ and $\lim_{x \rightarrow -\infty} R^*(x) + x = \mathbb{E}\nu$. We will show that R^* is convex, which is equivalent to the convexity of the epigraph \mathcal{E} of R^* . Pick two points $(x_1, y_1), (x_2, y_2) \in \mathcal{E}$. Then there exist measures $\theta_1, \theta_2 \in A_\mu^\nu$ such that $R_{\theta_1}(x_1) \leq y_1$ and $R_{\theta_2}(x_2) \leq y_2$. Using Lemma 3.1 once more, we get that $\theta := \theta_1 \wedge \theta_2 \in A_\mu^\nu$ and $R_\theta(x_i) \leq y_i, i = 1, 2$. Therefore, the whole segment with endpoints (x_1, y_1) and (x_2, y_2) lies in the epigraph of R_θ and hence in \mathcal{E} . This implies that R^* is a call function, and the associated measure has to be $S_\nu(\mu)$. Also, we can therefore conclude that the infimum in (3.13) is attained for all x .

Now assume that (3.12) is wrong. Since all functions appearing in (3.12) are right-continuous, there must then exist an open interval (a, b) where (3.12) does not hold, i.e. $R^*(x) > R_\nu(x)$ and $(R^*)'(x) \in (R'_\mu(x - \epsilon), R'_\mu(x + \epsilon))$ for all $x \in (a, b)$.

Case 1: There exists an open interval $I \subseteq (a, b)$ where R^* is strictly convex. Then we can pick $x_1 \in I$ and $h_1 > 0$ such that $x_1 + h_1 \in I$ and such that the tangent

$$P_1(x) := R^*(x_1) + (R^*)'(x_1)(x - x_1), \quad x \in [x_1, x_1 + h_1]$$

satisfies $R_\nu(x) < P_1(x) < R^*(x)$ for $x \in (x_1, x_1 + h_1]$. Also, since $(R^*)'(x_1) > R'_\mu(x_1 - \epsilon)$ and since R'_μ is right-continuous, we can choose h_1 small enough to guarantee $(R^*)'(x_1) \geq R'_\mu(x_1 + h_1 - \epsilon)$. Next pick $x_2 \in (x_1, x_1 + h_1)$, such that $R'_\mu(\cdot + \epsilon)$ is continuous at x_2 and set

$$P_2(x) := R^*(x_2) + (R^*)'(x_2)(x - x_2), \quad x \in [x_2 - h_2, x_2].$$

We can choose h_2 small enough to ensure that $R_\nu(x) < P_2(x) < R^*(x)$ and $(R^*)'(x_2) \leq R'_\mu(x_2 - h_2 + \epsilon)$. Also, if x_1 and x_2 are close enough together, then there is an intersection of P_1 and P_2 in (x_1, x_2) . Now the function

$$\tilde{R}(x) := \begin{cases} P_1(x) \vee P_2(x), & x \in [x_1, x_2], \\ R^*(x), & \text{otherwise,} \end{cases}$$

is a call function which is strictly smaller than R^* and satisfies $\tilde{R}'(x) \in [R'_\mu(x - \epsilon), R'_\mu(x + \epsilon)]$ for all $x \in \mathbb{R}$. This is a contradiction to (3.13). See Figure 2 for an illustration.

Case 2: If there is no open interval in (a, b) where R^* is strictly convex, then R^* has to be affine on some closed interval $I \subseteq (a, b)$ (see p. 7 in [30]). Therefore, there exist k, d in \mathbb{R} such that

$$R^*(x) = kx + d, \quad x \in I.$$

By Proposition 2.2 (ii), the slope k has to lie in the open interval $(-1, 0)$, since R^* is greater than R_ν on I . We set

$$\begin{aligned} a_1 &:= \sup \{ x \in \mathbb{R} : (R^*)'(x) < k \} > -\infty, \\ b_1 &:= \inf \{ x \in \mathbb{R} : (R^*)'(x) > k \} < \infty; \end{aligned}$$

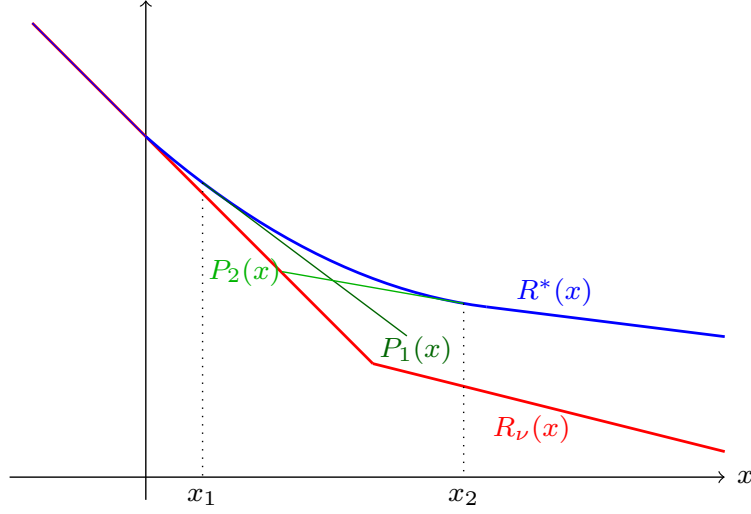


Figure 2: Case 1 of the proof of Theorem 3.8. If R^* is strictly convex, then we can deform it using two appropriate tangents, contradicting minimality of the associated measure.

the finiteness of these quantities follows from Proposition 2.2 (ii). From the convexity of R_ν and the fact that $R_\nu \leq R^*$, we get that $R^*(x) > R_\nu(x)$ for all $x \in (a_1, b_1)$, as well as $(R^*)'(x) > R'_\mu(x - \epsilon)$ for all $x \in (a_1, b)$ and $(R^*)'(x) < R'_\mu(x + \epsilon)$ for all $x \in (a, b_1)$. We now define lines P_1 and P_2 , with analogous roles as in Case 1. Their definitions depend on the behavior of $(R^*)'$ at a_1 and b_1 .

If $(R^*)'(a_1-) < k$, then we set $x_1 = a_1$ and $P_1(x) = R^*(x_1) + k_1(x - x_1)$ for $x \geq x_1$, with an arbitrary $k_1 \in ((R^*)'(x_1-), k)$; see Figure 3.

If, on the other hand, $(R^*)'(a_1-) = k$, then we can find $x_1 < a_1$ such that $R^*(x_1) > R_\nu(x_1)$ and $(R^*)'(x_1) > R'_\mu(x_1 - \epsilon)$. In this case we define

$$P_1(x) := R^*(x_1) + (R^*)'(x_1)(x - x_1), \quad x \geq x_1.$$

Similarly, if $(R^*)'(b_1) > k$, then we define $x_2 = b_1$ and $P_2(x) = R^*(x_2) + k_2(x - x_2)$ for $x \leq x_2$ and for $k_2 \in (k, (R^*)'(b_1))$, and otherwise we can find $x_2 > b_1$ such that $R^*(x_2) > R_\nu(x_2)$ and $(R^*)'(x_2) < R'_\mu(x_2 + \epsilon)$. We then set

$$P_2(x) := R^*(x_2) + (R^*)'(x_2)(x - x_2), \quad x \leq x_2.$$

We can choose $h_1, h_2 > 0$, $\tilde{d} < d$ and k_1, k_2 such that the function

$$\tilde{R}(x) := \begin{cases} P_1(x), & x \in [x_1, x_1 + h_1], \\ kx + \tilde{d}, & x \in [x_1 + h_1, x_2 - h_2], \\ P_2(x), & x \in [x_2 - h_2, x_2], \\ R^*(x), & \text{otherwise,} \end{cases}$$

is a call function which is strictly smaller than R^* but not smaller than R_ν . Also, if h_1 and h_2 are small enough we have that $\tilde{R}'(x) \in [R'_\mu(x - \epsilon), R'_\mu(x + \epsilon)]$ for all $x \in \mathbb{R}$, which is a contradiction to (3.13). \square

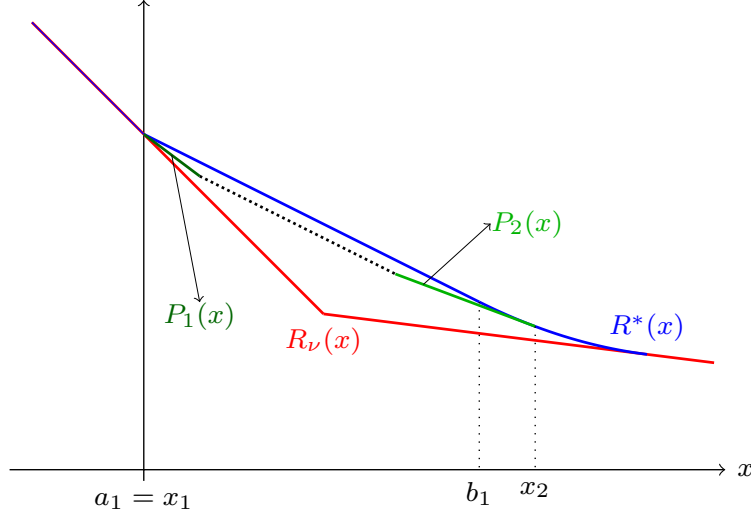


Figure 3: Case 2 of the proof of Theorem 3.8, with $(R^*)'(a_1-) < k$ and $(R^*)'(b_1) = k$.

In part (i) of Theorem 3.8, we showed that A_μ^ν has a least element. The weaker statement that it has an infimum follows from [21], p. 162; there it is shown that *any* subset of the lattice (\mathcal{M}_m, \leq_c) has an infimum. (The stated requirement that the set be bounded from below is always satisfied, as the Dirac delta δ_m is the least element of (\mathcal{M}_m, \leq_c) .) This infimum is, of course, given by the least element $S_\nu(\mu)$ that we found.

If $\nu = \delta_m$, then $S_\nu(\mu) = S(\mu)$, the least element from Proposition 3.2. In this case we have that

$$(R^*)'(x) = \begin{cases} R'_\mu(x - \epsilon), & x < x^*, \\ R'_\mu(x + \epsilon), & x \geq x^*, \end{cases}$$

where x^* is the unique solution of

$$m + R_\mu(x - \epsilon) - (\mathbb{E}\mu + \epsilon) = R_\mu(x + \epsilon).$$

The following corollary establishes an alternative representation of the inequality (3.12), which we will use to prove Theorem 3.5. Note that, in general, (3.12) has more than one solution, not all of which are call functions. However, R^* is always a solution.

Corollary 3.9. *Assume that the conditions from Theorem 3.8 hold and denote the call function of $S_\nu(\mu)$ with R^* . Then for all $x \in \mathbb{R}$ there exists $y \in \mathbb{R} \cup \{\pm\infty\}$ such that*

$$R^*(x) = R_\nu(y) - R_\mu(y + \epsilon\sigma) + R_\mu(x + \epsilon\sigma),$$

where $\sigma = \text{sgn}(y - x)$. Here and in the following we set $R(\infty) = 0$ for all call functions R and

$$R_1(-\infty \pm \epsilon) - R_2(-\infty \pm \epsilon) := \lim_{x \rightarrow -\infty} (R_1(x \pm \epsilon) - R_2(x \pm \epsilon)),$$

for call functions R_1 and R_2 .

Proof. By Theorem 3.8 we know that R^* is a solution of (3.12). Let x be an arbitrary real number. If $R^*(x) = R_\nu(x)$, then the above relation clearly holds for $y = x$. Otherwise, we have $R^*(x) > R_\nu(x)$, and one of the other two expressions on the left hand side of (3.12) must vanish at x . First we assume that $(R^*)'(x) = R'_\mu(x + \epsilon)$. Define

$$y := \inf\{z \geq x : (R^*)'(z) < R'_\mu(z + \epsilon)\}.$$

If $y < \infty$, then by definition $(R^*)'(y) < R'_\mu(y + \epsilon)$. By (3.12), we have $R^*(y) = R_\nu(y)$. It follows that

$$R^*(z) = R_\nu(y) - R_\mu(y + \epsilon) + R_\mu(z + \epsilon), \quad \text{for all } z \in [x, y].$$

If $y = \infty$, then this equation, i.e. $R^*(z) = R_\mu(z + \epsilon)$, $z \geq x$, also holds.

If, on the other hand, $(R^*)'(x) = R'_\mu(x - \epsilon)$, then we similarly define $y := \sup\{z \leq x : (R^*)'(z) > R'_\mu(z - \epsilon)\}$. If $y > -\infty$ then $(R^*)'(y-) > R'_\mu((y - \epsilon)-)$ and hence $R^*(y) = R_\nu(y)$ by (3.12). Therefore we can write

$$R^*(z) = R_\nu(y) - R_\mu(y - \epsilon) + R_\mu(z - \epsilon), \quad \text{for all } z \in [y, x].$$

If $y = -\infty$ then $(R^*)'(z) = R'_\mu(z - \epsilon)$ for all $z \leq x$. The above equation holds if we take the limit $y \rightarrow -\infty$ on the right hand side. \square

Corollary 3.10. *Using Proposition 3.2 and Theorem 3.8, for a given sequence of measures $(\mu_n)_{n \in \mathbb{N}}$ in \mathcal{M} , we inductively define the measures*

$$\theta_1 = S(\mu_1; m, \epsilon), \quad \theta_k = S_{\theta_{k-1}}(\mu_k), \quad k \geq 2,$$

if the sets

$$\left\{ \nu \in B^\infty(\mu_k, \epsilon) : \theta_{k-1} \leq_c \nu \right\}$$

are not empty. Then the following relation holds:

$$R_{\theta_n}(x) = R_{\theta_{n-1}}(y) - R_{\mu_n}(y + \epsilon\sigma) + R_{\mu_n}(x + \epsilon\sigma),$$

where $n \geq 2$, $y \in \mathbb{R} \cup \{\pm\infty\}$ depends on x and $\sigma = \text{sgn}(y - x)$.

Proof. The result follows by simply applying Theorem 3.8 and Corollary 3.9 with $\nu = \theta_{n-1}$ and $\mu = \mu_n$. \square

We can now prove Theorem 3.5, our main result. As in Strassen's theorem, the “if” direction is the more difficult one.

Proof of Theorem 3.5. Suppose that (3.9) holds for some $m \in I$ and all $N \in \mathbb{N}$, $x_1, \dots, x_N \in \mathbb{R}$. We will inductively construct a sequence $(P_n)_{n \in \mathbb{N}}$ of call functions, which will correspond to the measures $(\nu_n)_{n \in \mathbb{N}}$. Define $P_1 = R_1^{\min}(\cdot; m, \epsilon)$. For $N = 1$, (3.9) guarantees that $R_1^{\min}(x) \leq R_2^{\max}(x)$. Note that the continuity of the R_n guarantee that (3.9) also holds for $x_n \in \{\pm\infty\}$, if we set $\text{sgn}(\infty - \infty) = \text{sgn}(-\infty + \infty) = 0$. We can now use Theorem 3.8 together with Corollary 3.9, with $R_\nu = R_1^{\min}$ and $R_\mu = R_2$, to construct a call function P_2 , which satisfies

$$P_2(x) = R_1^{\min}(x_1) + R_2(x + \epsilon\sigma) - R_2(x_1 + \epsilon\sigma), \quad x \in \mathbb{R},$$

where $\sigma = \text{sgn}(x_1 - x)$, and x_1 depends on x . If we use (3.9) we get that

$$R_1^{\min}(x_1) + R_2(x + \epsilon\sigma) - R_2(x_1 + \epsilon\sigma) \leq R_n^{\max}(x; s, \epsilon), \quad n \geq 3, \quad x_1, x \in \mathbb{R}.$$

Hence $P_2(x) \leq R_n^{\max}(x)$ for all $x \in \mathbb{R}$ and for all $n \geq 3$. Now suppose that we have already constructed a finite sequence (P_1, \dots, P_N) such that $P_n \leq P_{n+1}$, $1 \leq n < N$, and such that $P_N \leq R_n^{\max}$ for all $x \in \mathbb{R}$ and for all $n \geq N+1$. Then by induction we know that for all $x \in \mathbb{R}$ there exists (x_1, \dots, x_{N-1}) such that

$$P_N(x) = R_1^{\min}(x_1) + \sum_{n=2}^{N-1} R_n(x_n + \epsilon \sigma_n) - R_n(x_{n-1} + \epsilon \sigma_n) + R_N(x + \epsilon \sigma_N) - R_N(x_{N-1} + \epsilon \sigma_N),$$

with $\sigma_N = \text{sgn}(x_{N-1} - x)$. In particular, we have that $P_N \leq R_{N+1}^{\max}$. We can therefore again use Corollary 3.9, with $R_\mu = R_{N+1}$ and $R_\nu = P_N$, to construct a call function P_{N+1} , such that

$$P_{N+1}(x) = R_1^{\min}(x_1) + \sum_{n=2}^N R_n(x_n + \epsilon \sigma_n) - R_n(x_{n-1} + \epsilon \sigma_n) + R_{N+1}(x + \epsilon \sigma_{N+1}) - R_{N+1}(x_N + \epsilon \sigma_{N+1}),$$

where $\sigma_{N+1} = \text{sgn}(x_N - x)$ and (x_1, \dots, x_N) depend on x . Assumption (3.9) guarantees that $P_{N+1} \leq R_n^{\max}$ for all $n \geq N+1$.

We have now constructed a sequence of call functions, such that $P_n \leq P_{n+1}$. Their associated measures, which we will denote with ν_n , satisfy $W^\infty(\mu_n, \nu_n) \leq \epsilon$ and $\nu_n \leq_c \nu_{n+1}$. Thus we have constructed a peacock with mean m .

Conversely, assume that $(\nu_n)_{n \in \mathbb{N}}$ is a peacock such that $W^\infty(\mu_n, \nu_n) \leq \epsilon$ and set $m = \mathbb{E}\nu_1$. Denote the call function of ν_n with P_n . We will show by induction that (3.9) holds. For $N = 1$ we have that

$$R_1^{\min}(x; m, \epsilon) \leq P_1(x) \leq P_2(x) \leq R_2^{\max}(x; m, \epsilon), \quad x \in \mathbb{R},$$

by Proposition 3.2.

For $N = 2$ and $x_1 \leq x_2$ we have that

$$\begin{aligned} R_1^{\min}(x_1; m, \epsilon) + R_2(x_2 - \epsilon) - R_2(x_1 - \epsilon) &\leq P_2(x_1) + \int_{x_1}^{x_2} R_2'(z - \epsilon) dz \\ &\leq P_2(x_1) + \int_{x_1}^{x_2} P_2'(z) dz \\ &= P_2(x_2) \leq P_3(x_2) \leq R_3^{\max}(x_2; m, \epsilon). \end{aligned}$$

Similarly, if $x_2 \leq x_1$,

$$\begin{aligned} R_1^{\min}(x_1; m, \epsilon) + R_2(x_2 + \epsilon) - R_2(x_1 + \epsilon) &\leq P_2(x_1) - \int_{x_2}^{x_1} R_2'(z + \epsilon) dz \\ &\leq P_2(x_1) - \int_{x_2}^{x_1} P_2'(z) dz \\ &= P_2(x_2) \leq P_3(x_2) \leq R_3^{\max}(x_2; m, \epsilon). \end{aligned}$$

If (3.9) holds for $N - 1$ and $x_{N-1} \leq x_N$, then

$$\begin{aligned} R_1^{\min}(x_1; m, \epsilon) + \sum_{n=2}^N (R_n(x_n + \epsilon \sigma_n) - R_n(x_{n-1} + \epsilon \sigma_n)) \\ \leq P_{N-1}(x_{N-1}) + R_N(x_N - \epsilon) - R_N(x_{N-1} - \epsilon) \\ \leq P_N(x_{N-1}) + \int_{x_{N-1}}^{x_N} P_N'(z) dz \\ \leq P_{N+1}(x_N) \leq R_{N+1}^{\max}(x_N; m, \epsilon). \end{aligned}$$

The case where $x_{N-1} \geq x_N$ can be dealt with similarly. □

Remark 3.11. In Theorem 3.5, it is actually not necessary that the balls centered at the measures μ_n are all of the same size. The theorem easily generalizes to the following result. For $m \in \mathbb{R}$, a sequence of non-negative numbers $(\epsilon_n)_{n \in \mathbb{N}}$, and a sequence of measures $(\mu_n)_{n \in \mathbb{N}}$ in \mathcal{M} , define

$$\begin{aligned} \Phi_N(x_1, \dots, x_N; m, \epsilon_1, \dots, \epsilon_{N+1}) &= R_1^{\min}(x_1; m, \epsilon_1) \\ &+ \sum_{n=2}^N (R_n(x_n + \epsilon_n \sigma_n) - R_n(x_{n-1} + \epsilon_n \sigma_n)) - R_{N+1}^{\max}(x_N; m, \epsilon_{N+1}), \\ N \in \mathbb{N}, \quad x_1, \dots, x_N \in \mathbb{R}, \end{aligned} \quad (3.14)$$

with σ_n defined in (3.6), and assume that

$$I := \bigcap_{n \in \mathbb{N}} [\mathbb{E}\mu_n - \epsilon_n, \mathbb{E}\mu_n + \epsilon_n]$$

is not empty. Then there exists a peacock $(\nu_n)_{n \in \mathbb{N}}$ such that

$$W^\infty(\mu_n, \nu_n) \leq \epsilon_n, \quad \text{for all } n \in \mathbb{N},$$

if and only if for some $m \in I$ and for all $N \in \mathbb{N}$, $x_1, \dots, x_N \in \mathbb{R}$, we have

$$\Phi_N(x_1, \dots, x_N; m, \epsilon_1, \dots, \epsilon_{N+1}) \leq 0.$$

To prove this result, one simply has to replace ϵ by ϵ_n in the preceding proof.

Remark 3.12. If a probability metric is comparable with the infinity Wasserstein distance, then our Theorem 3.5 implies a corresponding result about that metric (but, of course, not an “if and only if” condition). Denote by W^p the p -Wasserstein distance ($p \geq 1$), defined by

$$W^p(\mu, \nu) = \inf \left(\mathbb{E}[|X - Y|^p] \right)^{1/p}, \quad \mu, \nu \in \mathcal{M}.$$

The infimum is taken over all probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and random pairs (X, Y) with marginals given by μ and ν . Clearly, we have that for all $\mu, \nu \in \mathcal{M}$ and $p \geq 1$

$$W^\infty(\mu, \nu) \geq W^p(\mu, \nu).$$

Hence, given a sequence $(\mu_n)_{n \in \mathbb{N}}$, (3.9) is a sufficient condition for the existence of a peacock $(\nu_n)_{n \in \mathbb{N}}$, such that $W^p(\mu_n, \nu_n) \leq \epsilon$ for all $n \in \mathbb{N}$. But since the balls with respect to W^p are in general strictly larger than the balls with respect to W^∞ , we cannot expect (3.9) to be necessary.

4 Strassen’s theorem for the infinity Wasserstein distance: continuous time

In this section we will formulate a version of Theorem 3.5 for continuous index sets. We generalize the definition of Φ_N from (3.5) as follows. For finite sets $\mathcal{T} = \{t_1, \dots, t_{N+1}\} \subseteq [0, 1]$ with $t_1 < t_2 < \dots < t_{N+1}$, we set

$$\begin{aligned} \Phi_{\mathcal{T}}(x_1, \dots, x_N; m, \epsilon) &= R_{t_1}^{\min}(x_1; m, \epsilon) \\ &+ \sum_{n=2}^N \left(R_{t_n}(x_n + \epsilon \sigma_n) - R_{t_{n-1}}(x_n + \epsilon \sigma_n) \right) - R_{t_{N+1}}^{\max}(x_N; m, \epsilon). \end{aligned} \quad (4.1)$$

Here, $R_{t_1}^{\min}$ is the call function of $S(\mu_{t_1}; m, \epsilon)$, $R_{t_{N+1}}^{\max}$ is the call function of $T(\mu_{t_{N+1}}; m, \epsilon)$, and $\sigma_n = \text{sgn}(x_{n-1} - x_n)$ depends on x_{n-1} and x_n . Using $\Phi_{\mathcal{T}}$, we can now formulate a necessary and sufficient condition for the existence of a peacock within ϵ -distance. The continuity assumption (4.2) occurs in the proof in a natural way; we do not know to which extent it can be relaxed.

Theorem 4.1. *Assume that $(\mu_t)_{t \in [0,1]}$ is a family of measures in \mathcal{M} such that*

$$I := \bigcap_{t \in [0,1]} [\mathbb{E}\mu_t - \epsilon, \mathbb{E}\mu_t + \epsilon]$$

is not empty and such that

$$\lim_{s \uparrow t} F_{\mu_s} = F_{\mu_t}, \quad t \in [0, 1], \quad (4.2)$$

pointwise on \mathbb{R} . Then there exists a peacock $(\nu_t)_{t \in [0,1]}$ with

$$W^\infty(\mu_t, \nu_t) \leq \epsilon, \quad \text{for all } t \in [0, 1],$$

if and only if there exists $m \in I$ such that for all finite sets $\mathcal{T} = \{t_1, \dots, t_{N+1}\} \subset \mathbb{Q} \cap [0, 1]$ with $t_1 < t_2 < \dots < t_{N+1}$, and for all $x_1, \dots, x_N \in \mathbb{R}$ we have that

$$\Phi_{\mathcal{T}}(x_1, \dots, x_N; m, \epsilon) \leq 0. \quad (4.3)$$

In this case it is possible to choose $\mathbb{E}\nu_t = m$ for all $t \in [0, 1]$.

Proof. By Theorem 3.5, condition (4.3) is clearly necessary for the existence of such a peacock. In order to show that it is sufficient, we will first construct ν_q for $q \in \mathbb{Q} \cap [0, 1]$. Therefore fix $m \in I$ such that (4.3) holds and fix $q = \frac{s}{r} \in \mathbb{Q} \cap [0, 1]$. We will define a sequence of measures $(\nu_q^{(n)})_{n \in \mathbb{N}}$ as follows (recall the notation from Theorem 3.8): for fixed $n \in \mathbb{N}$ set $\theta_1^{(n)} = S_{\mu_0}(\mu_{\frac{1}{sn}})$ and $\theta_k^{(n)} = S_{\theta_{k-1}^{(n)}}(\mu_{\frac{k}{sn}})$, where $k = 2, \dots, rn$. Then

$$\nu_q^{(n)} := \theta_{rn}^{(n)}.$$

Condition (4.3) guarantees that $\nu_q^{(n)}$ exists. Denote the call function of $\nu_q^{(n)}$ with R_n . Then we have that

$$R_{S(\mu_q; m, \epsilon)} \leq R_n \leq R_{n+1} \leq R_{T(\mu_q; m, \epsilon)}, \quad n \in \mathbb{N}, \quad (4.4)$$

and thus the bounded and increasing sequence (R_n) converges pointwise to a function R . As a limit of decreasing convex functions R is also decreasing and convex and together with (4.4) we see that R is a call function with $\lim_{x \rightarrow -\infty} R(x) + x = m$. Therefore R can be associated to a measure $\nu_q \in \mathcal{M}_m$.

Next, we will show that $\nu_q \in B^\infty(\mu_q, \epsilon)$. From the convexity of the R_n we get that

$$\begin{aligned} R'(x) &= \lim_{h \downarrow 0} \lim_{n \rightarrow \infty} \frac{R_n(x+h) - R_n(x)}{h} \\ &\geq \lim_{h \downarrow 0} \lim_{n \rightarrow \infty} R'_n(x+h) \\ &\geq \lim_{h \downarrow 0} \lim_{n \rightarrow \infty} R'_{\mu_q}(x+h-\epsilon) = R'_{\mu_q}(x-\epsilon), \end{aligned}$$

and similarly

$$\begin{aligned}
R'(x) &= \lim_{h \downarrow 0} \lim_{n \rightarrow \infty} \frac{R_n(x+h) - R_n(x)}{h} \\
&\leq \lim_{h \downarrow 0} \lim_{n \rightarrow \infty} R'_n(x) \\
&\leq \lim_{n \rightarrow \infty} R'_{\mu_q}(x+\epsilon) = R'_{\mu_q}(x+\epsilon),
\end{aligned}$$

thus $W^\infty(\nu_q, \mu_q) \leq \epsilon$.

Now for $p, q \in \mathbb{Q} \cap [0, 1]$ we want to show that $\nu_p \leq_c \nu_q$. We will first illustrate the idea for $p = \frac{1}{3}$ and $q = \frac{1}{2}$. Recall that ν_p was defined via an approximating sequence $(\nu_p^{(n)})_{n \in \mathbb{N}}$. An easy observation reveals that $\nu_p^{(2)} \leq_c \nu_q^{(3)}$: Indeed, $\nu_p^{(2)}$ is defined to be the smallest element in $B^\infty(\mu_p, \epsilon) \cap \mathcal{M}_m$ which dominates $S_{\mu_0}(\mu_{\frac{1}{6}})$, and $\nu_q^{(3)}$ is defined to be the smallest element in $B^\infty(\mu_q, \epsilon) \cap \mathcal{M}_m$ which dominates $\nu_p^{(2)}$. With similar arguments we can show that $\nu_p^{(4)} \leq_c \nu_q^{(6)}$, or more generally $\nu_p^{(2^k)} \leq_c \nu_q^{(3 \cdot 2^{k-1})}$, for all $k \in \mathbb{N}$.

For general $p, q \in \mathbb{Q} \cap [0, 1]$ with $p = \frac{r_1}{s_1}$ and $q = \frac{r_2}{s_2}$, we pick subsequences $(l_k)_{k \in \mathbb{N}}$ and $(n_k)_{k \in \mathbb{N}}$ such that $l_k s_1 = n_k s_2$ for all $k \in \mathbb{N}$. Then clearly $\nu_p^{(l_k)} \leq_c \nu_q^{(n_k)} \leq_c \nu_q$ for all $k \in \mathbb{N}$ and therefore $\nu_p \leq_c \nu_q$. We have shown that $(\nu_t)_{t \in \mathbb{Q} \cap [0, 1]}$ is a peacock.

The next step is to define measures ν_t for $t \notin \mathbb{Q} \cap [0, 1]$. Therefore pick such a t and an increasing sequence $q_n \in \mathbb{Q} \cap [0, 1]$ which converges to t . Similar reasoning as before shows that the sequence $(R_{\nu_{q_n}})_{n \in \mathbb{N}}$ converges pointwise to a call function, and we define ν_t to be the associated measure. Then clearly $\mathbb{E}\nu_t = m$. Furthermore, using the continuity of the distribution functions, we get that

$$\begin{aligned}
R'_{\nu_t}(x) &= \lim_{h \downarrow 0} \lim_{n \rightarrow \infty} \frac{R_{\nu_{q_n}}(x+h) - R_{\nu_{q_n}}(x)}{h} \\
&\geq \lim_{h \downarrow 0} \lim_{n \rightarrow \infty} R'_{\nu_{q_n}}(x+h) \\
&\geq \lim_{h \downarrow 0} \lim_{n \rightarrow \infty} R'_{\mu_{q_n}}(x+h-\epsilon) \\
&= \lim_{h \downarrow 0} R'_{\mu_t}(x+h-\epsilon) = R'_{\mu_t}(x-\epsilon),
\end{aligned}$$

and similarly we see that $R'_{\nu_t}(x) \leq R'_{\mu_t}(x+\epsilon)$. We have shown that $\nu_t \in B^\infty(\mu_t, \epsilon)$ for all $t \in [0, 1]$.

From the definition of ν_t we have that $\nu_q \leq_c \nu_t$ for $q < t, q \in \mathbb{Q} \cap [0, 1]$ and $\nu_t \leq_c \nu_p$ for $p > t, p \in \mathbb{Q} \cap [0, 1]$. This implies $\nu_s \leq_c \nu_t$ for all $0 \leq s \leq t \leq 1$, thus $(\nu_t)_{t \in [0, 1]}$ is a peacock with mean m . \square

5 Strassen's theorem for the stop-loss distance

For two measures $\mu, \nu \in \mathcal{M}$ we define the stop-loss distance as

$$d^{\text{SL}}(\mu, \nu) = \sup_{x \in \mathbb{R}} |R_\mu(x) - R_\nu(x)|.$$

We will denote closed balls with respect to d^{SL} with B^{SL} . In the following proposition, we use the same notation for least elements as in the case of the infinity Wasserstein distance; no confusion should arise.

Proposition 5.1. *Given $\epsilon > 0$, a measure $\mu \in \mathcal{M}$ and $m \in [\mathbb{E}\mu - \epsilon, \mathbb{E}\mu + \epsilon]$, there exists a unique measure $S(\mu) \in B^{\text{SL}}(\mu, \epsilon) \cap \mathcal{M}_m$, such that*

$$S(\mu) \leq_c \nu, \quad \text{for all } \nu \in B^{\text{SL}}(\mu, \epsilon) \cap \mathcal{M}_m.$$

The call function of $S(\mu)$ is given by

$$R_{S(\mu)}^{\min}(x) = R_{S(\mu)}(x) = (m - x)^+ \vee (R_\mu(x) - \epsilon). \quad (5.1)$$

To highlight the dependence on ϵ and m we will sometimes write $S(\mu; m, \epsilon)$ or $R_\mu^{\min}(\cdot; m, \epsilon)$.

Proof. It is easy to check that $R_{S(\mu)}$ defines a call function, and by (ii) of Proposition 2.2 we have that

$$\begin{aligned} \mathbb{E}R_{S(\mu)} &= \lim_{x \rightarrow -\infty} R_{S(\mu)}(x) + x \\ &= \lim_{x \rightarrow -\infty} (m \vee (R_\mu(x) + x - \epsilon)) \\ &= m \vee (\mathbb{E}\mu - \epsilon) = m. \end{aligned}$$

The rest is clear. \square

Remark 5.2. The set $B^{\text{SL}}(\mu, \epsilon) \cap \mathcal{M}_m$ does not contain a greatest element. To see this, take an arbitrary $\nu \in B^{\text{SL}}(\mu, \epsilon) \cap \mathcal{M}_m$ and define $x_0 \in \mathbb{R}$ as the unique solution of $R_\nu(x) = \frac{1}{2}\epsilon$. Then for $n \in \mathbb{N}$ define new call functions

$$R_n(x) = \begin{cases} (x - x_0) \frac{R_\nu(x_0 + n) - R_\nu(x_0)}{n} + R_\nu(x_0), & x \in [x_0, x_0 + n], \\ R_\nu(x), & \text{otherwise.} \end{cases}$$

It is easy to check that R_n is indeed a call function and the associated measures θ_n lie in $B^{\text{SL}}(\mu, \epsilon) \cap \mathcal{M}_m$. Furthermore, from the convexity of R_ν we can deduce that $R_\nu \leq R_n \leq R_{n+1}$, and hence $\nu \leq_c \theta_n \leq_c \theta_{n+1}$. The call functions R_n converge to a function R which is not a call function since $R(x) = R_\nu(x_0) = \frac{\epsilon}{2}$ for all $x \geq x_0$. Therefore no greatest element can exist.

However it is true that a measure ν is in $B^{\text{SL}}(\mu, \epsilon)$ if and only if $R_\mu^{\min}(\cdot; \mathbb{E}\nu, \epsilon) \leq R_\nu \leq R_\mu + \epsilon$.

Theorem 5.3. *Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{M} such that*

$$I := \bigcap_{n \in \mathbb{N}} [\mathbb{E}\mu_n - \epsilon, \mathbb{E}\mu_n + \epsilon],$$

is not empty. Denote with $(R_n)_{n \in \mathbb{N}}$ the corresponding call functions. Then there exists a peacock $(\nu_n)_{n \in \mathbb{N}}$ such that

$$d^{\text{SL}}(\mu_n, \nu_n) \leq \epsilon, \quad n \in \mathbb{N}, \quad (5.2)$$

if and only if for some $m \in I$

$$R_k^{\min}(x; m, \epsilon) \leq R_n(x) + \epsilon, \quad \text{for all } k \leq n \text{ and } x \in \mathbb{R}. \quad (5.3)$$

Here R_k^{\min} denotes the call function of $S(\mu_k; m, \epsilon)$. In this case it is possible to choose $\mathbb{E}\nu_1 = m$.

Proof. Suppose (5.3) holds for $m \in I$. We will define the ν_n via their call functions P_n . Therefore we set $P_1(x) = R_1^{\min}(x; m, \epsilon)$ and

$$P_n(x) = \max \left\{ P_{n-1}(x), R_n^{\min}(x; m, \epsilon) \right\}, \quad n \geq 2. \quad (5.4)$$

It is easily verified that P_n is a call function and satisfies

$$R_n^{\min}(x) \leq P_n(x) \leq R_n(x) + \epsilon, \quad x \in \mathbb{R}, \quad (5.5)$$

and therefore ν_n , the measure associated to P_n , satisfies $\nu_n \in B^{\text{SL}}(\mu_n, \epsilon)$. Furthermore $P_n \leq P_{n+1}$, and thus $(\nu_n)_{n \in \mathbb{N}}$ is a peacock with mean m .

Now assume that $(\nu_n)_{n \in \mathbb{N}}$ is a peacock such that $d^{\text{SL}}(\mu_n, \nu_n) \leq \epsilon$. We will denote the call function of ν_n with P_n and set $m = \mathbb{E}\nu_1 \in I$. Then for $k \leq n$ and $x \in \mathbb{R}$ we get with Proposition 5.1

$$R_k^{\min}(x; m, \epsilon) \leq P_k(x) \leq P_n(x) \leq R_n(x) + \epsilon.$$

□

Note that (5.3) trivially holds for $k = n$. Moreover, unwinding the recursive definition (5.4) and using (5.1), we see that P_n has the explicit expression

$$P_n(x) = \max\{(m - x)^+, R_1(x) - \epsilon, \dots, R_n(x) - \epsilon\}, \quad x \in \mathbb{R}, n \in \mathbb{N}.$$

The following proposition shows that the peacock from Theorem 5.3 is never unique.

Proposition 5.4. *In the setting of Theorem 5.3, suppose that (5.3) holds. Then there are infinitely many peacocks satisfying (5.2).*

Proof. Define P_n as in the proof of Theorem 5.3, and fix $x_0 \in \mathbb{R}$ with $P_1(x_0) < \epsilon$. For arbitrary $c \in (0, 1)$, we define

$$G(x) = \begin{cases} P_1(x_0), & x \leq x_0, \\ P_1(x_0) + cP_1'(x_0)(x - x_0), & x \geq x_0. \end{cases}$$

Thus, in a right neighborhood of x_0 , the graph of G is a line that lies above P_1 . We then put $\tilde{P}_n = P_n \vee G$, for $n \in \mathbb{N}$. It is easy to see that (\tilde{P}_n) is an increasing sequence of call functions with mean m , and thus defines a peacock. Moreover, we have

$$\tilde{P}_n \leq (R_n + \epsilon) \vee G \leq R_n + \epsilon,$$

by (5.5) and the fact that $G \leq \epsilon$. The lower estimate $\tilde{P}_n \geq P_n \geq R_n - \epsilon$ is also obvious. □

Theorem 5.3 easily extends to continuous index sets.

Theorem 5.5. *Assume that $(\mu_t)_{t \in [0,1]}$ is a family of measures in \mathcal{M} such that*

$$I := \bigcap_{t \in [0,1]} [\mathbb{E}\mu_t - \epsilon, \mathbb{E}\mu_t + \epsilon]$$

is not empty. Denote the call function of μ_t with R_t . Then there exists a peacock $(\nu_t)_{t \in [0,1]}$ with

$$d^{\text{SL}}(\mu_t, \nu_t) \leq \epsilon, \quad \text{for all } t \in [0, 1],$$

if and only if there exists $m \in I$ such that for all $0 \leq s < t \leq 1$ we have that

$$R_s^{\min}(x; m, \epsilon) \leq R_t(x) + \epsilon, \quad \text{for all } x \in \mathbb{R}. \quad (5.6)$$

Here R_s^{\min} denotes the call function of $S(\mu_s; m, \epsilon)$. In this case it is possible to choose $\mathbb{E}\nu_1 = m$.

Proof. If (5.6) holds for $m \in I$ we set

$$P_t(x) = \sup_{s \leq t} R_s^{\min}(x; m, \epsilon), \quad t \in [0, 1].$$

Then P_t is a call function which satisfies $R_t^{\min}(x; m, \epsilon) \leq P_t(x) \leq R_t(x) + \epsilon$ for $x \in \mathbb{R}$. The rest can be done as in the proof of Theorem 5.3. □

6 Lévy distance and Prokhorov distance: preliminaries

We will begin with the definition the Lévy distance and the Prokhorov distance. For further information concerning these metrics, their properties and their relations to other metrics, we refer the reader to [17] (p.27 ff). The Lévy distance is a metric on the set of all measures on \mathbb{R} , defined as

$$d^L(\mu, \nu) = \inf \left\{ h > 0 : F_\mu(x - h) - h \leq F_\nu(x) \leq F_\mu(x + h) + h, \forall x \in \mathbb{R} \right\}.$$

Its importance is partially due to the fact that d^L metrizes weak convergence of measures on \mathbb{R} . The Prokhorov distance is a metric on measures on an arbitrary separable metric space (S, ρ) . For measures μ, ν on S it can be written as

$$d^P(\mu, \nu) = \inf \left\{ h > 0 : \nu(A) \leq \mu(A^h) + h, \text{ for all closed sets } A \subseteq S \right\},$$

where $A^h = \{x \in S : \inf_{a \in A} \rho(x, a) \leq h\}$. The Prokhorov distance is often referred to as a generalization of the Lévy metric, since d^P metrizes weak convergence on any separable metric space. Note, though, that d^L and d^P do not coincide when $(S, \rho) = (\mathbb{R}, |\cdot|)$, as shown in the following example.

Example 6.1. Let $\epsilon = \frac{1}{8}$, μ be the uniform distribution on $[0, 1]$, and ν be the uniform distribution on $[2\epsilon, 1 - 2\epsilon]$. Then it is easy to check that $d^L(\mu, \nu) \leq \epsilon$. Also we have that

$$F_\mu\left(\frac{1}{4} - \epsilon\right) - \epsilon = F_\nu\left(\frac{1}{4}\right),$$

hence $d^L(\mu, \nu) = \epsilon$. Next, we will show that the Prokhorov distance of μ and ν is larger than $\frac{1}{6}$, and hence not equal to the Lévy distance. Consider the closed set $B = [2\epsilon, 1 - 2\epsilon]$. Then $\nu(B) = 1$, and the inequality

$$\begin{aligned} 1 &\leq \mu(B^h) + h \\ &= \mu([2\epsilon - h, 1 - 2\epsilon + h]) + h = 1 - 4\epsilon + 3h \end{aligned}$$

is true for all $h \geq \frac{1}{6}$, and therefore $d^P(\mu, \nu) \geq \frac{1}{6}$.

It is easy to see that the Prokhorov distance of two measures on \mathbb{R} is an upper bound for the Lévy distance. See [17] p.36; we include the simple proof for completeness.

Lemma 6.2. *Let μ and ν be two probability measures on \mathbb{R} . Then*

$$d^L(\mu, \nu) \leq d^P(\mu, \nu).$$

Proof. We set $d^P(\mu, \nu) = \epsilon$. Then for any $x \in \mathbb{R}$ and all $n \in \mathbb{N}$ we have that

$$\begin{aligned} F_\nu(x) = \nu((-\infty, x]) &\leq \mu\left(\left(-\infty, x + \epsilon + \frac{1}{n}\right]\right) + \epsilon + \frac{1}{n} \\ &= F_\mu\left(x + \epsilon + \frac{1}{n}\right) + \epsilon + \frac{1}{n}, \end{aligned}$$

and by the symmetry of d^P the above relation also holds with μ and ν interchanged. This implies that $d^L(\mu, \nu) \leq \epsilon$. \square

7 Modified Lévy distance and Prokhorov distance

We will first define slightly different distances d_p^L and d_p^P on the set of probability measures on \mathbb{R} , which in general are not metrics in the classical sense (recall the remark before Proposition 2.5). These distances are useful for two reasons: First, it will turn out that balls with respect to d_p^L and d_p^P can always be written as balls w.r.t. d_p^L and d_p^P , see Lemma 7.1. Second, the function d_p^P has a direct link to minimal distance couplings which are especially useful for applications, see Proposition 7.3.

For $p \in [0, 1]$ we define

$$d_p^L(\mu, \nu) := \inf \left\{ h > 0 : F_\mu(x - h) - p \leq F_\nu(x) \leq F_\mu(x + h) + p, \forall x \in \mathbb{R} \right\} \quad (7.1)$$

and

$$d_p^P(\mu, \nu) := \inf \left\{ h > 0 : \nu(A) \leq \mu(A^h) + p, \text{ for all closed sets } A \subseteq S \right\}. \quad (7.2)$$

It is easy to show that $d_p^P(\mu, \nu) = d_p^P(\nu, \mu)$ (see e.g. Proposition 1 in [10]). Note that $d_p(\mu, \nu) = 0$ does not imply that $\mu = \nu$. We will refer to d_p^L as the modified Lévy distance, and to d_p^P as the modified Prokhorov distance. The following Lemma explains the connection between the Lévy distance d^L and the modified Lévy distance d_p^L , resp. the Prokhorov distance d^P and the modified Prokhorov distance d_p^P .

Lemma 7.1. *Let $\mu \in \mathcal{M}$. Then for every $\epsilon \in [0, 1]$ we have that*

$$B^L(\mu, \epsilon) = B_\epsilon^L(\mu, \epsilon), \quad \text{and} \quad B^P(\mu, \epsilon) = B_\epsilon^P(\mu, \epsilon).$$

Proof. For $\nu \in \mathcal{M}$, the assertion $\nu \in B^P(\mu, \epsilon)$ is equivalent to

$$\mu(A) \leq \nu(A^{\epsilon+\delta}) + \epsilon + \delta, \quad \delta > 0, \quad A \subseteq \mathbb{R} \text{ closed}, \quad (7.3)$$

whereas $\nu \in B_\epsilon^P(\mu, \epsilon)$ means that

$$\mu(A) \leq \nu(A^{\epsilon+\delta}) + \epsilon, \quad \delta > 0, \quad A \subseteq \mathbb{R} \text{ closed}. \quad (7.4)$$

Obviously, (7.4) implies (7.3). Now suppose that (7.3) holds, and let $\delta \downarrow 0$. Notice that $A^{\epsilon+\delta_1} \subseteq A^{\epsilon+\delta_2}$ for $\delta_1 \leq \delta_2$. The continuity of ν then gives

$$\mu(A) \leq \nu(A^\epsilon) + \epsilon \leq \nu(A^{\epsilon+\delta}) + \epsilon \quad \delta > 0, \quad A \subseteq \mathbb{R} \text{ closed},$$

and thus $B^P(\mu, \epsilon) = B_\epsilon^P(\mu, \epsilon)$.

Replacing A by intervals $(-\infty, x]$ for $x \in \mathbb{R}$ in (7.3) and (7.4) proves that $B^L(\mu, \epsilon) = B_\epsilon^L(\mu, \epsilon)$. \square

Similarly to Lemma 6.2 we can show that the modified Lévy distance of two measures never exceeds the modified Prokhorov distance.

Lemma 7.2. *Let μ and ν be two probability measures on \mathbb{R} and let $p \in [0, 1]$. Then*

$$d_p^L(\mu, \nu) \leq d_p^P(\mu, \nu).$$

Proof. We set $\epsilon = d_p^P(\mu, \nu)$. Then for any $x \in \mathbb{R}$ and all $n \in \mathbb{N}$ we have that

$$\begin{aligned} F_\nu(x) &= \nu((-\infty, x]) \leq \mu\left(\left(-\infty, x + \epsilon + \frac{1}{n}\right]\right) + p \\ &= F_\mu\left(x + \epsilon + \frac{1}{n}\right) + p, \end{aligned}$$

and by the symmetry of d^P the above relation also holds with μ and ν interchanged. This implies that $d_p^L(\mu, \nu) \leq \epsilon$. \square

The following result was first proved by Strassen and was then extended by Dudley [10, 31]. It explains the connection of d_p^P to minimal distance couplings.

Proposition 7.3. *Given measures μ, ν on \mathbb{R} , $p \in [0, 1]$, and $\epsilon > 0$ there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with random variables $X \sim \mu$ and $Y \sim \nu$ such that*

$$\mathbb{P}(|X - Y| > \epsilon) \leq p, \quad (7.5)$$

if and only if

$$d_p^P(\mu, \nu) \leq \epsilon. \quad (7.6)$$

8 Strassen's theorem for Prokhorov distance and Lévy distance

In this section we will prove variants of Strassen's theorem, first for the modified Prokhorov distance and later on for the modified Lévy distance, the Prokhorov distance, and the Lévy distance. It turns out that Problem 2.3 always has a solution for these distances, regardless of the size of ϵ . In the following we denote the quantile function of a measure $\mu \in \mathcal{M}$ with G_μ , i.e.

$$G_\mu(p) = \inf \{x \in \mathbb{R} : F_\mu(x) \geq p\}, \quad p \in [0, 1].$$

Proposition 8.1. *Let $\mu \in \mathcal{M}$, $p \in (0, 1]$, and $m \in \mathbb{R}$. Then the set*

$$B_p^P(\mu, 0) \cap \mathcal{M}_m$$

is not empty. Moreover, this set contains at least one measure with bounded support.

Proof. The statement is clear for $p = 1$, and so we focus on $p \in (0, 1)$. Given a measure μ we set $I = [G_\mu(\frac{p}{4}), G_\mu(1 - \frac{p}{4})]$. We will first define a measure η with bounded support which lies in $B_p^P(\mu, 0)$, and then we will modify it to obtain a measure θ with mean m . We set

$$F_\eta(x) := \begin{cases} 0, & x < G_\mu(\frac{p}{4}), \\ F_\mu(x), & x \in I, \\ 1, & x \geq G_\mu(1 - \frac{p}{4}), \end{cases}$$

which is clearly a distribution function of a measure η . Note that η has finite support, so in particular η has finite mean. Next we define

$$\theta = \left(1 - \frac{p}{2}\right)\eta + \frac{p}{2}\delta_w,$$

where w is chosen such that $\mathbb{E}\theta = m$. Since η has bounded support, we can deduce that θ also has bounded support. Now for every closed set $A \subseteq \mathbb{R}$ we have that

$$\begin{aligned} \theta(A) &\leq \left(1 - \frac{p}{2}\right)\eta(A) + \frac{p}{2} \\ &\leq \left(1 - \frac{p}{2}\right)\eta(A \cap \text{int}(I)) + p \\ &\leq \mu(A) + p, \end{aligned}$$

where $\text{int}(I)$ denotes the interior of I . For the last inequality, note that μ and η are equal on $\text{int}(I)$. The last equation implies that $\theta \in B_p^P(\mu, 0) \cap \mathcal{M}_m$. \square

Note that in Proposition 8.1 it is not important that μ has finite mean. The statement is true for all measures on \mathbb{R} . The same is true for all subsequent results.

Proposition 8.2. *Let $\nu \in \mathcal{M}$ be a measure with bounded support and $p \in (0, 1)$. Then for all measures $\mu \in \mathcal{M}$ there exists a measure $\theta \in B_p^P(\mu, 0)$ with bounded support such that $\nu \leq_c \theta$.*

Proof. Fix $\mu, \nu \in \mathcal{M}$ and $p \in (0, 1)$, and set $m = \mathbb{E}\nu$. Then, by Proposition 8.1, there is a measure $\theta_0 \in B_{p/2}^P(\mu, 0) \cap \mathcal{M}_m$ which has bounded support. For $n \in \mathbb{N}$ we define

$$\theta_n = \left(1 - \frac{p}{2}\right)\theta_0 + \frac{p}{4}\delta_{m-n} + \frac{p}{4}\delta_{m+n}.$$

These measures have bounded support and mean m . Furthermore, for $A \subseteq \mathbb{R}$ closed, we have

$$\begin{aligned} \theta_n(A) &\leq \left(1 - \frac{p}{2}\right)\theta_0(A) + \frac{p}{2} \\ &\leq \theta_0(A) + \frac{p}{2} \leq \mu(A) + p, \quad n \in \mathbb{N}, \end{aligned}$$

and hence $\nu_n \in B_p^P(\mu, 0)$ for all $n \in \mathbb{N}$. Now observe that for all $n \in \mathbb{N}$ and $x \in (m - n, m + n)$ we have

$$R_{\theta_n}(x) = \left(1 - \frac{p}{2}\right)R_{\theta_0}(x) + \frac{p}{4}(m + n - x), \quad (8.1)$$

which tends to infinity as n tends to infinity. Therefore there has to exist $n_0 \in \mathbb{N}$ such that $\nu \leq_c \theta_{n_0}$. \square

In Proposition 8.2 it is important that $p > 0$. For $p = 0$ the limit in 8.1 is finite.

Theorem 8.3. *Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{M} , $\epsilon > 0$. and $p \in (0, 1]$. Then, for all $m \in \mathbb{R}$ there exists a peacock $(\nu_n)_{n \in \mathbb{N}}$ with mean m such that*

$$d_p^P(\mu_n, \nu_n) \leq \epsilon.$$

Proof. If $p = 1$ then $B_p^P(\mu, 0)$ contains all probability measures on \mathbb{R} , which is easily seen from the definition of d_p^P , and the result is trivial. So we consider the case $p < 1$. Since $B_p^P(\mu, 0) \subseteq B_p^P(\mu, \epsilon)$, it suffices to prove the statement for $\epsilon = 0$. By Proposition 8.1, there exists a measure $\nu_1 \in B_p^P(\mu_1, 0) \cap \mathcal{M}_m$ with bounded support. By Proposition 8.2 there exists a measure $\nu_2 \in B_p^P(\mu_2, 0)$ such that $\nu_1 \leq_c \nu_2$. Since ν_2 has again finite support, we can proceed inductively to finish the proof. \square

Setting $\epsilon = p \in (0, 1]$ in the previous result, we obtain the following corollary.

Corollary 8.4. *Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{M} and $\epsilon > 0$. Then, for all $m \in \mathbb{R}$ there exists a peacock $(\nu_n)_{n \in \mathbb{N}}$ with mean m such that*

$$d^P(\mu_n, \nu_n) \leq \epsilon.$$

Proof. By Lemma 7.1 we have that $B^P(\mu, \epsilon) = B_\epsilon^P(\mu, \epsilon)$ for all $\mu \in \mathcal{M}$ and $\epsilon \in [0, 1]$. The result now easily follows from Theorem 8.3. \square

Since balls with respect to the modified Prokhorov metric are larger than balls with respect to the Lévy metric, we get the following corollary.

Theorem 8.5. *Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{M} , $\epsilon > 0$, and $p \in (0, 1]$. Then, for all $m \in \mathbb{R}$ there exists a peacock $(\nu_n)_{n \in \mathbb{N}}$ with mean m such that*

$$d_p^L(\mu_n, \nu_n) \leq \epsilon.$$

In particular, there exists a peacock $(\nu_n)_{n \in \mathbb{N}}$ with mean m such that

$$d^L(\mu_n, \nu_n) \leq \epsilon.$$

Proof. Fix $\epsilon > 0$ and $p \in (0, 1]$, and let $(\nu_n)_{n \in \mathbb{N}}$ be the peacock from Theorem 8.3 resp. Corollary 8.4. Then by Lemma 7.2 resp. Lemma 6.2, we have that $\nu_n \in B_p^L(\mu_n, \epsilon)$ resp. $\nu_n \in B^L(\mu_n, \epsilon)$ for all $n \in \mathbb{N}$. \square

Remark 8.6. For $\mu \in \mathcal{M}$, $\epsilon \geq 0$, $p \in (0, 1)$, and $m \in \mathbb{R}$ the set $B_p^L \cap \mathcal{M}_m$ contains a least element with respect to \leq_c . Its call function is given by

$$R_{S(\mu)} = \max\{P_\mu(x), Q_\mu(x), (m - x)^+\}, \quad x \in \mathbb{R},$$

where

$$\begin{aligned} P_\mu(x) &= \left(R_\mu(x - \epsilon) - px - (R_\mu(a - \epsilon) - pa) + (m - a)^+ \right) \mathbf{1}_{[a, \infty)}(x), \\ Q_\mu(x) &= \left(R_\mu(x + \epsilon) + px - (R_\mu(b + \epsilon) + pb) \right) \mathbf{1}_{(-\infty, b]}(x), \end{aligned}$$

and $a = G_\mu(p) + \epsilon$, $b = G_\mu(1 - p) - \epsilon$. See I. C. Gülüm's forthcoming PhD thesis for details.

9 Conclusion

Given a family of measures $(\mu_t)_{t \in T}$ on \mathbb{R} with finite means, we derived conditions for the existence of a peacock $(\nu_t)_{t \in T}$ within a certain distance to the given family. We formulated necessary and sufficient conditions for the infinity Wasserstein distance and the stop-loss distance, and showed that such a peacock always exists if we measure distance with the Lévy metric or the Prokhorov metric. In particular, we get the following result, which is a simple corollary of Theorem 3.5, Theorem 8.3 and Corollary 8.4. Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of probability measures on \mathbb{R} with finite means, $\epsilon > 0$, and $p \in (0, 1]$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a stochastic process $(X_n)_{n \in \mathbb{N}}$ whose marginal laws are given by $(\mu_n)_{n \in \mathbb{N}}$ and a martingale $(M_n)_{n \in \mathbb{N}}$ such that:

- (i) $\mathbb{P}(|X_n - M_n| > \epsilon) \leq p$ for all $n \in \mathbb{N}$,
- (ii) $\mathbb{P}(|X_n - M_n| > \epsilon) \leq \epsilon$ for all $n \in \mathbb{N}$,
- (iii) $\mathbb{P}(|X_n - M_n| > \epsilon) = 0$ for all $n \in \mathbb{N}$ if and only if condition 3.9 holds,
- (iv) $\mathbb{P}(X_n = M_n) = 1$ for all $n \in \mathbb{N}$ if and only if $(\mu_n)_{n \in \mathbb{N}}$ is a peacock.

Notice that (iv) is simply Strassen's theorem. In future work we hope to prove similar statements for other metrics, e.g. the p -Wasserstein distance W^p ($p \geq 1$).

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